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A New Method for the Calculation of the Emergent Distributions for the Anisotropic Slab Albedo Problem*

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A method is presented for the calculation of the emergent particle distributions of the albedo problem in one-speed transport theory with arbitrary anisotropic scattering. From a singular eigenfunction expansion of the solution of the albedo problem and the orthogonality properties of these eigenfunctions, a pair of coupled Fredholm integral equations of the second kind are obtained for the reflected and transmitted distributions. These integral equations are readily decoupled, and can be approximated by a set of linear equations from which very accurate numerical solutions are obtained.

I. INTRODUCTION

One of the most important problems in one-speed particle transport theory for media with plane and azimuthal symmetry is the slab albedo problem. In this problem a monodirectional beam of particles is incident on one surface of a homogeneous, isotropic, source-free slab of thickness τ, which is surrounded by vacuum. Inside the slab all particles are assumed to travel with the same speed. The mean number of secondaries per collision is denoted by c, and the scattering function (or phase function) is assumed to be expanded in a finite sum of Legendre polynomials of the scattering angle. Then, if distance x is measured in units of the mean free path and direction by the cosine μ of the angle between the velocity vector and the positive x axis, the angular distribution ψ of the particles inside the slab satisfies [1] the equation

$$\mu \frac{\partial \psi}{\partial x} + \psi(x, \mu) = \frac{c}{2} \sum_{n=0}^{N} b_n P_n(\mu) \int_{-1}^{1} d\mu' \, P_n(\mu') \, \psi(x, \mu') \qquad (1\text{-}1)$$

for $0 \leqslant x \leqslant \tau$ and $-1 \leqslant \mu \leqslant 1$, with the boundary conditions

$$\psi(0, \mu) = \delta(\mu - \mu_0) \qquad 0 < \mu, \quad \mu_0 < 1, \qquad (1\text{-}2)$$

$$\psi(\tau, \mu) = 0 \qquad -1 < \mu < 0. \qquad (1\text{-}3)$$

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The b_n's are the coefficients of the $N + 1$ term Legendre polynomial expansion of the scattering function.

The calculation of the distribution ψ is encountered in many areas of mathematical physics, particularly in neutron transport theory and radiative transfer. The slab albedo problem, besides being important in its own right, is perhaps the most fundamental finite medium transport problem since the solutions to all other slab problems (e.g., Green's function) can be expressed in terms of the slab albedo problem solution and known infinite medium solutions [2]. A closed form solution to the slab albedo problem, even for the simplest case of isotropic scattering, is not known. In many applications (e.g., neutron shielding problems and diffuse reflection from planetary atmospheres) the complete solution is not required, but rather only the particle distributions at the slab surfaces are sought, i.e., $R(\mu) \equiv \psi(0, -\mu)$, $\mu > 0$ and $T(\mu) = \psi(\tau, \mu)$, $\mu > 0$. Even for these emergent distributions no closed form solution has been found, although many computational schemes have been suggested, most of which are based on the calculation of the X and Y functions of Chandrasekhar [3]. These X and Y functions, from which (together with some associated polynomials) the emergent albedo distributions are evaluated, can be found by numerical solution of various pairs of coupled equations—nonlinear integral equations [4], nonlinear integrodifferential equations [5], linear singular integral equations [6], and regular Fredholm equations [7, 8].

The purpose of this paper is to present a new method for calculation of the emergent distributions of the slab albedo problem. This method has the advantage that it allows computation of the emergent distributions directly without first calculating auxilary functions such as the X and Y functions. The reflected and transmitted distributions R and T are shown to satisfy a pair of coupled regular Fredholm integral equations which are readily decoupled. The numerical solution of these equations is easily obtained and gives very accurate results even for high degrees of scattering anisotropy.

The Fredholm equations for the reflected and transmitted particle distributions are derived from Case's singular eigenfunction expansion technique [1, 9] as applied to the anisotropic slab albedo problem. With this technique the albedo problem solution is expanded in terms of a complete set of eigenfunctions of Eq. (1-1). In the usual application of Case's technique, singular integral equations (which for the slab albedo problem can be reduced to Fredholm equations) for the expansion coefficients are obtained. To evaluate the emergent distributions, these equations for the expansion coefficients must be solved numerically, and then the original singular eigenfunction expansion is evaluated at the slab surfaces. Such numerical evaluation is far from being trivial especially if the scattering function is highly anisotropic [10, 11]. In the present method, use of the orthogonality properties of these eigenfunctions allows the expansion coefficients to be expressed directly in terms of the emergent distributions. Substitution of these results into the eigenfunction expansion of the slab albedo solution at the slab surfaces yields a pair of coupled Fredholm equations for the emergent distributions.

In the next section a brief review is presented of the singular eigenfunction expansion technique as applied to the slab albedo problem. Then in Sections III and IV the Fredholm equations for the emergent distributions are derived for the cases $c < 1$ and $c = 1$, respectively. Finally the numerical solution of these Fredholm equations is discussed.

II. Singular Eigenfunction Expansion of ψ

The complete solution of the slab albedo problem can be uniquely expanded as [1, 9]

$$\psi(x, \mu) = \sum_{j=1}^{M} a_{+j}\phi(\nu_j, \mu)\, e^{-x/\nu_j} + \sum_{j=1}^{M} a_{-j}\phi(-\nu_j, \mu)\, e^{x/\nu_j} + \int_{-1}^{1} d\nu\, A(\nu)\, \phi(\nu, \mu)\, e^{-x/\nu}, \tag{2-1}$$

where the $a_{\pm j}$ and $A(\nu)$ are the expansion coefficients and ϕ and ν are the eigenfunctions and eigenvalues of the homogeneous transport equation. Explicitly for $|\nu| > 1$ the eigenfunctions are given by

$$\phi(\pm\nu_j, \mu) = \frac{c\nu_j}{2} \frac{D(\nu_j, \pm\mu)}{\nu_j \mp \mu} \qquad j = 1 \cdots M, \tag{2-2}$$

where the M pairs $(1 \leqslant M \leqslant N + 1)$ of discrete eigenvalues, $\pm\nu_j$, are given by the dispersion relation

$$1 - \frac{c\nu}{2} \int_{-1}^{1} d\mu \frac{D(\mu, \mu)}{\nu - \mu} = 0. \tag{2-3}$$

The function D is given by

$$D(\nu, \mu) = \sum_{n=0}^{N} b_n h_n(\nu)\, P_n(\mu), \tag{2-4}$$

where the polynomial h_n is defined by the recurrence relation

$$(n + 1)\, h_{n+1}(\nu) - \nu(2n + 1 - b_n c)\, h_n(\nu) + n h_{n-1}(\nu) = 0, \qquad n = 0, 1, 2,... \tag{2-5}$$

and $h_0 \equiv 1$.

For $\nu \in (-1, 1)$ the eigenfunctions are given by

$$\phi(\nu, \mu) = \frac{c\nu}{2} \frac{D(\nu, \mu)}{\nu - \mu} + \lambda(\nu)\, \delta(\nu - \mu), \tag{2-6}$$

where the generalized functions $(\nu - \mu)^{-1}$ and $\delta(\nu - \mu)$ are defined by the functionals

$$\langle(\nu - \mu)^{-1}, \Phi(\mu)\rangle = \oint_{-1}^{1} d\mu \frac{\Phi(\mu)}{\nu - \mu} \tag{2-7}$$

and

$$\begin{aligned} \langle\delta(\nu - \mu), \Phi(\mu)\rangle &= \Phi(\nu) \qquad \text{if} \quad \nu \in [-1, 1] \\ &= 0 \qquad \text{if} \quad \nu \notin [-1, 1], \end{aligned} \tag{2-8}$$

respectively, for all test functions Φ. The symbol $\oint$ refers to integration in the Cauchy principal value sense. The function λ is defined as

$$\lambda(\nu) = 1 - \frac{c\nu}{2} \oint_{-1}^{1} d\mu \frac{D(\nu, \mu)}{\nu - \mu}. \tag{2-9}$$

The eigenfunctions ϕ satisfy an orthogonality relation which symbolically may be written as

$$\int_{-1}^{1} d\mu\, \mu\phi(\nu, \mu)\, \phi(\nu', \mu) = 0 \qquad \text{if} \quad \nu \neq \nu' \quad \text{for} \quad \nu, \nu' \in (-1, 1) \text{ or } \pm\nu_j, \tag{2-10}$$

$$\int_{-1}^{1} d\mu\, \mu\phi^2(\pm\nu_j, \mu) = \pm N_j^{-1}, \tag{2-11}$$

$$\int_{-1}^{1} d\mu\, \mu\phi(\nu, \mu)\, \phi(\nu', \mu) = N^{-1}(\nu)\, \delta(\nu - \nu'), \tag{2-12}$$

where

$$N_j^{-1} \equiv \frac{c\nu_j^2}{2} \int_{-1}^{1} d\mu \frac{D^2(\nu_j, \mu)}{(\nu_j - \mu)^2}, \tag{2-13}$$

$$N(\nu) \equiv \{\nu[\lambda^2(\nu) + ((c\pi\nu/2)\, D(\nu, \nu))^2]\}^{-1}. \tag{2-14}$$

Because the continuum eigenfunctions are linear functionals, Eqs. (2-10) and (2-12) are actually abbreviations for the statement

$$\int_{-1}^{1} d\mu\, \mu\phi(\nu, \mu) \int_{a}^{b} d\nu' f(\nu')\, \phi(\nu', \mu) = \begin{cases} f(\nu)\, N^{-1}(\nu) & \text{if} \quad \nu \in (a, b) \\ 0 & \text{if} \quad \nu \notin (a, b), \end{cases} \tag{2-15}$$

where $-1 \leqslant a < b \leqslant 1$ and f is any continuous function.

III. Derivation of Fredholm Equations for $c < 1$

From the above orthogonality relations, expressions for the expansion coefficients of Eq. (2-1) can be obtained in terms of the emergent distributions. Set $x = 0$ in Eq. (2-1), multiply through by $\mu\phi(\nu_j, \mu)$, integrate over μ, and use the orthogonality properties together with boundary condition (1-2) to obtain

$$a_{+j} = N_j \left\{ \mu_0 \phi(\nu_j, \mu_0) - \int_0^1 d\mu\, \mu\phi(-\nu_j, \mu)\, R(\mu) \right\}. \tag{3-1}$$

Similarly, by multiplying Eq. (2-1) by $\phi(\nu, \mu)$, $\nu > 0$ instead of $\phi(\nu_j, \mu)$ one obtains

$$A(\nu) = N(\nu) \left\{ \mu_0 \phi(\nu, \mu_0) - \int_0^1 d\mu\, \mu\phi(-\nu, \mu)\, R(\mu) \right\}, \qquad \nu > 0. \tag{3-2}$$

To calculate a_{-j} set $x = \tau$ in Eq. (2-1), multiply through by $\mu\phi(-\nu_j, \mu)$, integrate over μ, and use the orthogonality relations plus Eq. (1-3). The result is

$$a_{-j} = -N_j e^{-\tau/\nu_j} \int_0^1 d\mu\, \mu\phi(-\nu_j, \mu)\, T(\mu). \tag{3-3}$$

Similarly $A(-\nu)$ is given by

$$A(-\nu) = -N(\nu)\, e^{-\tau/\nu} \int_0^1 d\mu\, \mu\phi(-\nu, \mu)\, T(\mu). \tag{3-4}$$

Substitution of these results into the expansion (2-1) gives the complete solution $\psi(x, \mu)$ in terms of the reflected and transmitted distributions. In particular, this result can be evaluated for the reflected and transmitted distributions by setting $x = 0$ and τ, respectively. The result is a pair of coupled integral equations, namely,

$$\begin{aligned} R(\mu) = {} & \mu_0 \sum_{j=1}^{M} N_j \phi(\nu_j, \mu_0)\, \phi(-\nu_j, \mu) + \mu_0 \int_0^1 d\nu\, N(\nu)\, \phi(\nu, \mu_0)\, \phi(-\nu, \mu) \\ & - \int_0^1 d\mu'\, \mu' K(\mu, \mu')\, R(\mu') - \int_0^1 d\mu'\, \mu' G(\mu, \mu')\, T(\mu'), \quad \mu > 0 \end{aligned} \tag{3-5}$$

and

$$\begin{aligned} T(\mu) = {} & \mu_0 \sum_{j=0}^{M} N_j \phi(\nu_j, \mu_0)\, \phi(\nu_j, \mu)\, e^{-\tau/\nu_j} + \mu_0 \int_0^1 d\nu\, N(\nu)\, \phi(\nu, \mu_0)\, \phi(\nu, \mu)\, e^{-\tau/\nu} \\ & - \int_0^1 d\mu'\, \mu' G(\mu, \mu')\, R(\mu') - \int_0^1 d\mu'\, \mu' K(\mu, \mu')\, T(\mu'), \quad \mu > 0, \end{aligned} \tag{3-6}$$

where the kernels K and G are defined as

$$K(\mu, \mu') \equiv \sum_{j=1}^{M} N_j \phi(-\nu_j, \mu') \phi(-\nu_j, \mu) + \int_0^1 d\nu\, N(\nu)\, \phi(-\nu, \mu')\, \phi(-\nu, \mu) \qquad (3\text{-}7)$$

and

$$G(\mu, \mu') \equiv \sum_{j=1}^{M} N_j \phi(-\nu_j, \mu') \phi(\nu_j, \mu)\, e^{-\tau/\nu_j} + \int_0^1 d\nu\, N(\nu)\, \phi(-\nu, \mu')\, \phi(\nu, \mu)\, e^{-\tau/\nu}. \qquad (3\text{-}8)$$

Equations (3-5) and (3-6) are not very amenable to numerical solution, since the transmitted distribution T is comprised of a continuous collided contribution $\mathscr{T}(\mu)$ and the singular uncollided source, $e^{-\tau/\mu_0}\delta(\mu - \mu_0)$. The singularity introduced by the uncollided source is manifested by the second term on the right of Eq. (3-6). However, the singularity in the above equations can be removed by rewriting these equations solely in terms of the collided transmitted distribution $\mathscr{T}$, viz.,

$$\mathscr{T}(\mu) \equiv T(\mu) - \delta(\mu - \mu_0)\, e^{-\tau/\mu_0}. \qquad (3\text{-}9)$$

The result obtained is

$$\begin{aligned} \mathscr{T}(\mu) + \delta(\mu - \mu_0)\, e^{-\tau/\mu_0} = \mu_0 e^{-\tau/\mu_0} \Big\{ & \sum_{j=1}^{M} N_j \phi(\nu_j, \mu_0)\, \phi(\nu_j, \mu) \\ & + \int_0^1 d\nu\, N(\nu)\, \phi(\nu, \mu_0)\, \phi(\nu, \mu) \\ & - \sum_j N_j \phi(-\nu_j, \mu_0)\, \phi(-\nu_j, \mu) \\ & - \int_0^1 d\nu\, N(\nu)\, \phi(-\nu, \mu_0)\, \phi(-\nu, \mu) \Big\} \\ & + g(\mu) - \int_0^1 d\mu'\, \mu' G(\mu, \mu')\, R(\mu') \\ & - \int_0^1 d\mu'\, \mu' K(\mu, \mu')\, \mathscr{T}(\mu'), \end{aligned} \qquad (3\text{-}10)$$

where

$$\begin{aligned} g(\mu) \equiv \mu_0 & \sum_{j=1}^{M} N_j \phi(\nu_j, \mu_0)\, \phi(\nu_j, \mu) [e^{-\tau/\nu_j} - e^{-\tau/\mu_0}] \\ & + \mu_0 \int_0^1 d\nu\, N(\nu)\, \phi(\nu, \mu_0)\, \phi(\nu, \mu) [e^{-\tau/\nu} - e^{-\tau/\mu_0}]. \end{aligned} \qquad (3\text{-}11)$$

The term in the braces on the right of Eq. (3-10) is simply the singular eigenfunction expansion of the functional $\delta(\mu - \mu_0)/\mu_0$ [12], and hence

$$\mathscr{T}(\mu) = g(\mu) - \int_0^1 d\mu'\, \mu' G(\mu, \mu')\, R(\mu') - \int_0^1 d\mu'\, \mu' K(\mu, \mu')\, \mathscr{T}(\mu'), \quad \mu > 0. \tag{3-12}$$

Substitution of Eq. (3-9) into Eq. (3-5) gives

$$R(\mu) = k(\mu) - \int_0^1 d\mu'\, \mu' K(\mu, \mu')\, R(\mu') - \int_0^1 d\mu'\, \mu' G(\mu, \mu')\, \mathscr{T}(\mu'), \quad \mu > 0, \tag{3-13}$$

where

$$\begin{aligned} k(\mu) \equiv \mu_0 \sum_{j=1}^{M} N_j \left\{ \phi(\nu_j, \mu_0)\, \phi(-\nu_j, \mu) - \phi(-\nu_j, \mu_0)\, \phi(\nu_j, \mu) \exp\left[-\frac{\tau}{\mu_0} - \frac{\tau}{\nu_j} \right] \right\} \\ + \mu_0 \int_0^1 d\nu\, N(\nu) \left\{ \phi(\nu, \mu_0)\, \phi(-\nu, \mu) - \phi(-\nu, \mu_0)\, \phi(\nu, \mu) \exp\left[-\frac{\tau}{\mu_0} - \frac{\tau}{\nu} \right] \right\}. \end{aligned} \tag{3-14}$$

The functions $k(\mu)$, $g(\mu)$, $K(\mu, \mu')$ and $G(\mu, \mu')$ for $\mu, \mu' \in (0, 1)$ are continuous and thus Eqs. (3-12) and (3-13) are a pair of regular Fredholm equations of the second kind for the emergent distributions. In Section V a method for the numerical solution of these Fredholm equations is discussed.

IV. Derivation of Fredholm Equations for $c = 1$

The situation when c approaches unity is of particular interest in many physical situations. This limiting situation is known as the "nonabsorbing" case in transport theory and the "conservative" case in radiative transfer. The results of the previous section must be modified since the two discrete eigenvalues $\pm\nu_1$ coalesce at infinity as c approaches unity, and thus the expansion of the albedo problem solution given by Eq. (2-1) is no longer valid. However, the eigensolution expansion technique can still be used by introducing two particular solutions of the homogeneous transport equation. Two such solutions are

$$\psi_1 \equiv \tfrac{1}{2} \tag{4-1}$$

and

$$\psi_2 \equiv \tfrac{1}{2}(x + 3\mu/(b_1 - 3)). \tag{4-2}$$

With these two particular solutions in place of the eigensolutions

$\phi(\pm\nu_1, \mu) \exp[\mp x/\nu_1]$, the albedo problem solution can be expanded uniquely as [9]

$$\psi(x, \mu) = a_1\psi_1 + a_{-1}\psi_{-1} + \sum_{j=2}^{M} a_j\phi(\nu_j, \mu) e^{-x/\nu_j} + \sum_{j=2}^{M} a_{-j}\phi(-\nu_j, \mu) e^{x/\nu_j}$$
$$+ \int_{-1}^{1} d\nu\, A(\nu)\, \phi(\nu, \mu)\, e^{-x/\nu}. \tag{4-3}$$

For the case $c = 1$, the remaining eigensolutions satisfy two integral relations

$$\int_{-1}^{1} d\mu\, \mu\phi(\nu, \mu) = \int_{-1}^{1} d\mu\, \mu^2\phi(\nu, \mu) = 0, \qquad \nu \in (-1, 1) \text{ or } \nu_j, \quad j = 2 \cdots M. \tag{4-4}$$

These results imply the eigensolutions are orthogonal to the two particular solutions. Hence the coefficients $a_{\pm j}$, $j = 2 \cdots M$ and $A(\nu)$ can be found in the same manner as in the previous section, and are given by the same expressions, namely Eqs. (3-1)–(3-4). The coefficients $a_{\pm 1}$ can be evaluated in terms of the emergent distributions by setting $x = 0$ and $x = \tau$ in Eq. (4-3), multiplying by μ and μ^2, and integrating over μ. Then from the relationships of Eq. (4-4) one obtains

$$a_1 = \frac{1}{2}\left\{3 \int_0^1 d\mu\, \mu^2[R(\mu) + T(\mu)] - \tau(b_1 - 3) \int_0^1 d\mu\, \mu T(\mu) + 3\mu_0^2\right\} \tag{4-3}$$

and

$$a_{-1} = \frac{b_1 - 3}{2}\left\{\int_0^1 d\mu\, \mu[T(\mu) - R(\mu)] + \mu_0\right\}. \tag{4-6}$$

Substitution of these results for $a_{\pm j}$, $j = 1 \cdots M$ and $A(\nu)$ into Eq. (4-3) gives the complete albedo problem solution in terms of the reflected and emergent distributions. In particular, by setting $x = 0$ and $x = \tau$ in the resulting expression, coupled integral equations, completely analogous to Eqs. (3-5) and (3-6), are obtained for R and T. As before, if we analytically extract the uncollided delta function source from the R–T equations, a pair of equations which is much more amenable to numerical treatment results. For $c = 1$, the singular eigenfunction expansion of $\delta(\mu - \mu_0)$ is

$$\delta(\mu - \mu_0) = \mu_0 \left\{\frac{3}{2}(\mu + \mu_0) + \sum_{j=2}^{M} N_j[\phi(\nu_j, \mu_0)\, \phi(\nu_j, \mu) - \phi(-\nu_j, \mu_0)\, \phi(-\nu_j, \mu)]\right.$$
$$\left.\int_{-1}^{1} d\nu\, N(\nu)\, \phi(\nu, \mu_0)\, \phi(\nu, \mu)\right\}. \tag{4-7}$$

Using this expansion together with Eq. (3–9), the R–T equations reduced to a pair of coupled regular Fredholm equations of the second kind of exactly the same form as that of Eqs. (3-12) and (3-13). For the case $c = 1$, the functions k, g, K and G are defined as

$$k(\mu) \equiv \mu_0 \Big\{ \frac{3}{4} (\mu_0 - \mu)(1 + e^{-\tau/\mu_0}) - \frac{\tau}{4} (b_1 - 3) e^{-\tau/\mu_0} + \sum_{j=2}^{M} N_j \Big[\phi(\nu_j, \mu_0) \phi(-\nu_j, \mu) - \phi(-\nu_j, \mu_0) \phi(\nu_j, \mu) \exp\Big(-\frac{\tau}{\nu_j} - \frac{\tau}{\mu_0}\Big)\Big] + \int_{-1}^{1} d\nu\, N(\nu) \Big[\phi(\nu, \mu_0) \phi(-\nu, \mu) - \phi(-\nu, \mu_0) \phi(\nu, \mu) \exp\Big(-\frac{\tau}{\nu} - \frac{\tau}{\mu_0}\Big)\Big]\Big\}, \tag{4-8}$$

$$g(\mu) \equiv \mu_0 \Big\{ \frac{3}{4} (\mu_0 + \mu)(1 - e^{-\tau/\mu_0}) + \frac{\tau}{4} (b_1 - 3) + \sum_{j=2}^{M} N_j \phi(\nu_j, \mu_0) \phi(\nu_j, \mu) \times [e^{-\tau/\nu_j} - e^{-\tau/\mu_0}] + \int_{0}^{1} d\nu\, N(\nu) \phi(\nu, \mu_0) \phi(\nu, \mu) [e^{-\tau/\nu} - e^{-\tau/\mu_0}] \Big\}, \tag{4-9}$$

$$K(\mu, \mu') \equiv -\frac{3}{4} (\mu + \mu') + \sum_{j=2}^{M} N_j \phi(-\nu_j, \mu') \phi(-\nu_j, \mu) + \int_{0}^{1} d\nu\, N(\nu) \phi(-\nu, \mu') \phi(-\nu, \mu), \tag{4-10}$$

and

$$G(\mu, \mu') \equiv \frac{3}{4} (\mu - \mu') + \frac{\tau}{4} (b_1 - 3) + \sum_{j=2}^{M} N_j \phi(\nu_j, \mu) \phi(-\nu_j, \mu') e^{-\tau/\nu_j} + \int_{0}^{1} d\nu\, N(\nu) \phi(\nu, \mu) \phi(-\nu, \mu') e^{-\tau/\nu}. \tag{4-11}$$

V. Numerical Solution of The Fredholm Equations

(a) *Decoupling and Evaluation*

The Fredholm equations for the emergent distributions derived in the previous sections are readily decoupled. Define

$$W(\mu) \equiv R(\mu) + \mathscr{T}(\mu) \tag{5-1}$$

and

$$Z(\mu) \equiv R(\mu) - \mathscr{T}(\mu). \tag{5-2}$$

Addition and subtraction of Eqs. (3-12) and (3-13) give the following pair of uncoupled Fredholm equations for the W and Z functions:

$$W(\mu) = k(\mu) + g(\mu) - \int_0^1 d\mu' \, \mu'[K(\mu, \mu') + G(\mu, \mu')] \, W(\mu') \qquad (5\text{-}3)$$

and

$$Z(\mu) = k(\mu) - g(\mu) - \int_0^1 d\mu' \, \mu'[K(\mu, \mu') - G(\mu, \mu')] \, Z(\mu'). \qquad (5\text{-}4)$$

The numerical solution of this pair of integral equations can be obtained in a number of ways. The most direct method is to approximate the Fredholm integrals by numerical quadrature, and then to evaluate Eqs. (5-3) and (5-4) at the various quadrature abscissas, thereby obtaining a set of linear equations. These linear equations may be solved readily by standard techniques to yield values of W and Z at the quadrature abscissas. Once W and Z are known, the transmitted and reflected distributions are found from Eqs. (5-1) and (5-2).

To evaluate the coefficients and inhomogeneous terms in the linear equations which approximate the integral equations (5-3) and (5-4), it is necessary to evaluate the functions k, g, K and G at the quadrature abscissas. The evaluation of these functions will necessitate further use of numerical quadrature. Further, in the calculation of k, g and G principal value integrals are involved. These principal value integrals can be transformed for numerical purposes into ordinary integrals by observing, for any Hölder continuous function f,

$$⨍_0^1 d\nu \, \frac{f(\nu)}{\nu - \mu} = \int_0^1 d\nu \frac{f(\nu) - f(\mu)}{\nu - \mu} + f(\mu) \ln \left[\frac{1 - \mu}{\mu}\right], \qquad \mu \in (0, 1). \qquad (5\text{-}5)$$

Using this result together with the definitions of the eigenfunctions (Eqs. (2-2) and (2-6)), explicit expressions for k, g, K and G, suitable for numerical evaluation, can be obtained. In particular, for the nonconservative case ($c < 1$), there results

$$k(\mu) = \mu_0\{k_1(\mu) + k_2(\mu) + k_3(\mu)\}, \qquad (5\text{-}6)$$

$$g(\mu) = \frac{c^2\mu_0}{4} \sum_{j=1}^{M} \frac{\nu_j^2 N_j D(\nu_j , \mu_0) \, D(\nu_j , \mu)}{(\nu_j - \mu_0)(\nu_j - \mu)} \, [e^{-\tau/\nu_j} - e^{-\tau/\mu_0}] + \mu_0 g_1(\mu), \qquad (5\text{-}7)$$

$$K(\mu', \mu) = K(\mu, \mu') = \frac{c^2}{4} \left\{ \sum_{j=1}^{M} \frac{\nu_j^2 N_j D(-\nu_j , \mu) \, D(-\nu_j , \mu')}{(\nu_j + \mu)(\nu_j + \mu')} + \int_0^1 d\nu \, \frac{\nu^2 N(\nu) \, D(-\nu, \mu) \, D(-\nu, \mu')}{(\nu + \mu)(\nu + \mu')} \right\}, \qquad (5\text{-}8)$$

and

$$G(\mu, \mu') = \frac{c^2}{4} \sum_{j=1}^{M} \frac{\nu_j^2 N_j D(-\nu_j, \mu') D(\nu_j, \mu)}{(\nu_j + \mu')(\nu_j - \mu)} e^{-\tau/\nu_j} + G_1(\mu, \mu'), \tag{5-9}$$

where

$$k_1(\mu) = \frac{c^2}{4} \sum_{j=1}^{M} N_j \nu_j^2 \left[\frac{D(\nu_j, \mu_0) D(-\nu_j, \mu)}{(\nu_j - \mu_0)(\nu_j + \mu)} - \frac{D(-\nu_j, \mu_0) D(\nu_j, \mu)}{(\nu_j + \mu_0)(\nu_j - \mu)} \exp\left(-\frac{\tau}{\mu_0} - \frac{\tau}{\nu_j}\right)\right], \tag{5-10}$$

$$k_2(\mu) = \frac{c}{2} \left\{\int_0^1 d\nu \, \frac{h(\nu, \mu, \mu_0) - h(\mu_0, \mu, \mu_0)}{\nu - \mu_0} + \frac{\mu_0 N(\mu_0) D(-\mu_0, \mu) f(\mu_0)}{\mu + \mu_0}\right\}, \tag{5-11}$$

$$k_3(\mu) = -\frac{c}{2} e^{-\tau/\mu_0} \left\{\int_0^1 d\nu \, \frac{h(\nu, \mu_0, \mu) e^{-\tau/\nu} - h(\mu, \mu_0, \mu) e^{-\tau/\mu}}{\nu - \mu} + \frac{\mu N(\mu) D(-\mu, \mu_0) f(\mu) e^{-\tau/\mu}}{\mu + \mu_0}\right\}, \tag{5-12}$$

$$g_1(\mu) = \frac{c}{2} \int_0^1 d\nu \, \frac{\nu g_0(\nu) D(\nu, \mu) - \mu g_0(\mu) D(\mu, \mu)}{\nu - \mu} + g_0(\mu) f(\mu), \tag{5-13}$$

$$G_1(\mu, \mu') = \frac{c}{2} \left\{\int_0^1 d\nu \, \frac{h(\nu, \mu', \mu) e^{-\tau/\nu} - h(\mu, \mu', \mu) e^{-\tau/\mu}}{\nu - \mu} + \frac{\mu N(\mu) D(-\mu, \mu') f(\mu) e^{-\tau/\mu}}{\mu + \mu'}\right\}, \tag{5-14}$$

$$h(\nu, \mu, \mu_0) \equiv \frac{c\nu^2 N(\nu) D(-\nu, \mu) D(\nu, \mu_0)}{2(\nu + \mu)}, \tag{5-15}$$

$$g_0(\nu) \equiv \begin{cases} \dfrac{c}{2} \dfrac{\nu D(\nu, \mu_0) N(\nu)}{(\nu - \mu_0)} [e^{-\tau/\nu} - e^{-\tau/\mu_0}] & \nu \neq \mu_0 \\ \dfrac{c}{2} \dfrac{\tau D(\mu_0, \mu_0) N(\mu_0)}{\mu_0} e^{-\tau/\mu_0} & \nu = \mu_0, \end{cases} \tag{5-16}$$

$$f(\mu) \equiv \frac{c}{2} \mu D(\mu, \mu) \ln\left(\frac{1 - \mu}{\mu}\right) + \lambda(\mu), \tag{5-17}$$

and

$$\lambda(\mu) = 1 - \frac{c\mu}{2} \left\{\int_0^1 d\nu \, \frac{D(\mu, \nu) - D(\mu, \mu)}{\mu - \nu} + \int_0^1 d\nu \, \frac{D(-\mu, \nu) - D(\mu, \mu)}{\nu + \mu} + D(\mu, \mu) \ln\left(\frac{1 + \mu}{1 - \mu}\right)\right\}. \tag{5-18}$$

For the conservative case ($c = 1$), one has

$$k(\mu) = k'(\mu) + \frac{\mu_0}{4}\{3(\mu_0 - \mu)(1 + e^{-\tau/\mu_0}) - \tau(b_1 - 3)\, e^{-\tau/\mu_0}\}, \tag{5-19}$$

$$g(\mu) = g'(\mu) + \frac{\mu_0}{4}\{3(\mu_0 + \mu)(1 - e^{-\tau/\mu_0}) + \tau(b_1 - 3)\}, \tag{5-20}$$

$$K(\mu, \mu') = K'(\mu, \mu') - \frac{3}{4}(\mu + \mu'), \tag{5-21}$$

$$G(\mu, \mu') = G'(\mu, \mu') + \frac{3}{4}(\mu - \mu') + \frac{\tau}{4}(b_1 - 3), \tag{5-22}$$

where k', g', K' and G' are given by Eqs. (5-6)–(5-9) with the discrete spectrum summation operator $\sum_{j=1}^{M}$ in Eqs. (5-7)–(5-10) replaced by $\sum_{j=2}^{M}$. The numerical evaluation of the integrals in Eqs. (5-11)–(5-14) and (5-18) can be performed by using a quadrature approximation. However, because of the indeterminacy of the integrands in these equations when $\nu = \mu$, it is necessary that a different set of quadrature abscissas be used than that used for approximating the Fredholm integrals.

Finally, before the functions k, g, K and G can be evaluated, the constants N_j and ν_j and the functions D and N must first be obtained. The discrete eigenvalues ν_j and their multiplicity M may be calculated from the dispersion relation of Eq. (2-3) [11]. The function N is readily calculated from Eq. (2-14). The discrete normalization N_j can be found from numerical evaluation of Eq. (2-13) provided ν_j is sufficiently greater than unity. Often, however, some of the ν_j are only very slightly greater than unity, and much more accurate results may be obtained by using Eq. (2-13) in the form

$$N_j^{-1} = \frac{c^2\nu_j^2}{4}\left\{\int_0^1 d\mu\, \mu\, \frac{D^2(\nu_j, \mu) - D^2(\nu_j, 1)}{(\nu_j - \mu)^2} - \int_0^1 d\mu\, \mu\, \frac{D^2(-\nu_j, \mu)}{(\nu_j + \mu)^2} + D^2(\nu_j, 1)\left[\frac{1}{\nu_j - 1} + \ln\left(\frac{\nu_j - 1}{\nu_j}\right)\right]\right\}. \tag{5-23}$$

The evaluation of the function $D(\nu, \mu)$ for $|\nu|, |\mu| \leqslant 1$ is readily obtained directly from its definition, Eqs. (2-4) and (2-5). However, for $|\nu|$ appreciably greater than unity and for a large degree of anisotrophy N, it is exceedingly difficult to evaluate this function. That such difficulties should arise is not surprising since the n-th term in the sum of Eq. (2-4) is a polynomial in ν of degree n, and for large n (and $|\nu| > 1$) this term will be very large and vary rapidly with ν. Only for ν exceedingly close to the discrete eigenvalues ν_j do the large terms in the sum of Eq. (2-4) add together to produce a smooth and slowly varying function of μ. For very large N the accuracy required for ν_j and the ensuing evaluation of Eq. (2-4) is often greater than that available in a computer. This difficulty in evaluating D arises from the expansion of the scattering function $f(\mu, \mu')$ as

$$f(\mu, \mu') = \sum_{n=0}^{N} b_n P_n(\mu') P_n(\mu). \tag{5-24}$$

However, the function $D(\nu, \mu)$ for $|\nu| > 1$ can also be defined by the integral equation [10, 11]

$$D(\nu, \mu) = \frac{c\nu}{2} \int_{-1}^{1} d\mu' f(\mu, \mu') \frac{D(\nu, \mu')}{\nu - \mu'} \tag{5-25}$$

which has a nontrivial solution if and only if ν equals one of the eigenvalues ν_j. The evaluation of $D(\nu_j, \mu)$ from this equation is discussed in detail in Refs. [10, 11].

(b) *Discussion of Results*

A computer program has been written to calculate the emergent distributions from the Fredholm equations developed in the previous sections. Gaussian quadrature for the interval (0, 1) was used for evaluation of the Fredholm integrals (order m) and the integrals in the functions k, g, K and G (order n). This particular quadrature approximation is well suited for the present analysis, since the quadrature abscissas are clustered near the end points. The integrands in the definitions of k, g, K and G often tend to vary rapidly near the upper endpoint, while the functions R and $\mathscr{T}$ exhibit relatively large variations near $\mu = 0$. To study the effect of anisotropic scattering it is necessary to specify a scattering or phase function. For the present analysis the following fictitious function has been used [11]:

$$f^N(\mathbf{\Omega} \cdot \mathbf{\Omega}') = \frac{N+1}{2^N} (1 + \mathbf{\Omega} \cdot \mathbf{\Omega}')^N, \qquad N = 0, 1, 2, \ldots . \tag{5-26}$$

This forward scattering function, which resembles many scattering functions encountered in physical situations, becomes concentrated near $\mathbf{\Omega} \cdot \mathbf{\Omega}' = 1$ as N increases. Mathematically this function has the advantage that it can be expanded exactly in terms of the first $N + 1$ Legendre polynomials, namely,

$$f^N(\mu, \mu') \equiv \int_0^{2\pi} d\phi' f^N(\mathbf{\Omega} \cdot \mathbf{\Omega}') = \sum_{n=0}^{N} b_n{}^N P_n(\mu) P_n(\mu'), \tag{5-27}$$

where the coefficients $b_n{}^N$ are given by the recurrence relation

$$b_n{}^N = \frac{N+1}{2N} \left[\frac{n}{2n-1} b_{n-1}^{N-1} + b_n^{N-1} + \frac{n+1}{2n+3} b_{n+1}^{N-1} \right] \tag{5-28}$$

with $b_0{}^N = 1$ ($N = 0, 1, 2, \ldots$) and $b_n{}^N = 0$ if $n > N$.

TABLE I

Comparison of calculated values of reflected and transmitted distributions from a slab characterized by $c = 0.95$, $\tau = 1.0$ and $\mu_0 = 0.5$ for various degrees of anisotropy of the scattering function f^N

Method	$N = 0$		$N = 10$		$N = 30$		$N = 30$	
	$R(0.5)$	$\mathscr{T}(0.5)$	$R(0.5)$	$\mathscr{T}(0.5)$	$R(0.5)$	$\mathscr{T}(0.5)$	$R(0.5)$	$\mathscr{T}(0.5)$
Fred. Eqs. $m = 3$, $n = 4$	0.4947	0.3499	0.33	1.01				
Fred. Eqs. $m = 5$, $n = 6$	0.494417	0.349683	0.248	0.765	0.139	1.42		
Fred. Eqs. $m = 7$, $n = 8$	0.494412	0.349682	0.2440	0.7533	0.112	1.19	0.065	1.58
Fred. Eqs. $m = 15$, $n = 16$	0.4944034	0.3496825	0.24372	0.75243	0.10974	1.1693	0.05583	1.4401
Fred. Eqs. $m = 21$, $n = 30$	0.4944032	0.3496837	0.243703	0.752411	0.109718	1.16915	0.055811	1.43999
Fred. Eqs. $m = 21$, $n = 50$	0.4944033	0.3496840	0.2437008	0.7524076	0.1097157	1.169137	0.0558094	1.439986
SLABCODE [11]	0.4944033	0.3496842	0.2437000	0.7524061	0.109715	1.16913	0.055809	1.43992
ANISN [13]	0.4945	0.3508	0.243747	0.7539	0.1096	1.1711	0.0557	1.444

Many calculations have been performed for various values of τ, c and N using different orders of Gaussian quadrature. It was found that a higher order of quadrature n is necessary for evaluation of k, g, K and G compared to the order m needed for the Fredholm integrals. For given values of m and n, the accuracy of the results tend to increase as c and τ increase and N decreases. Table I presents a comparison of the reflected and transmitted distributions as calculated from the Fredholm equations of this paper for various orders of Gaussian quadrature, and various degrees of anisotropy. Also included in this table are values as calculated by the codes SLABCODE [11] (based on the singular eigenfunction solution of Kaper [9]) and ANISN [13] (based on a discrete ordinate calculation using DP_7 quadrature set). In Fig. 1 the calculated emergent distributions for a particular slab albedo problem are shown for various degrees of anisotropy.

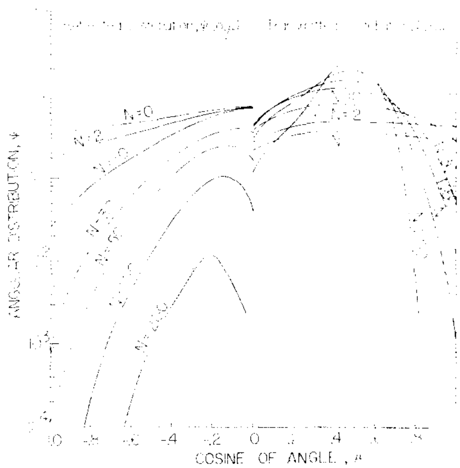

FIG. 1. The reflected and transmitted particle distributions from a slab with $c = 0.95$, $\tau = 1.0$ and $\mu_0 = 0.5$ for various degrees of anisotropy of the scattering function f^N.

To assess the accuracy of the present method for given m and n, the slab thickness was set to zero and the total albedo α and transmittance β defined as

$$\alpha = \int_0^1 d\mu\ \mu R(\mu) \qquad \text{and} \qquad \beta = \int_0^1 d\mu\ \mu \mathscr{T}(\mu) \tag{5-29}$$

were calculated. As τ increases, results generally increase in accuracy since the functions g and G (the most difficult to evaluate accurately) decrease in magnitude and have less of an effect on the computations. Thus, the amount α and β differ from zero for $\tau = 0$, gives an indication of the maximum error to be expected in calculations for a slab of finite thickness using the same quadrature sets. In Table II albedo and transmittance results are presented.

TABLE II

Calculated albedo and transmittance for a slab of zero thickness ($c = 0.95$, $\mu_0 = 0.5$) for various orders of anisotropy

	$m = 5, n = 6$		$m = 15, n = 16$		$m = 21, n = 50$	
N	α	β	α	β	α	β
0	−0.21(−3)	0.12(−3)	−0.21(−4)	0.17(−4)	−0.21(−5)	0.11(−5)
2	−0.24(−3)	0.10(−3)	−0.35(−4)	0.14(−4)	−0.24(−5)	0.96(−6)
5	0.73(−2)	−0.79(−2)	0.91(−3)	−0.30(−3)	0.33(−3)	−0.11(−3)
10	0.52(−2)	−0.14(−2)	0.38(−4)	−0.82(−5)	0.24(−3)	−0.49(−6)
20	0.42(−1)	−0.86(−2)	0.41(−3)	−0.11(−3)	0.20(−4)	−0.55(−5)
30	0.60(−1)	−0.15(−1)	−0.19(−5)	−0.73(−5)	−0.95(−6)	−0.24(−6)
40			0.11(−2)	−0.27(−3)	0.85(−4)	−0.20(−4)
50			0.57(−4)	−0.98(−5)	0.19(−5)	−0.20(−6)
100					0.62(−4)	−0.23(−4)

The time required for the code to calculate the emergent distributions for a particular problem varied greatly depending on how the functions $D(\pm\nu_j, \mu)$ were computed. If these functions were calculated directly from Eqs. (2-4) and (2-5) the problem, using the present technique, was calculated in about one fortieth of the time required by SLABCODE. However, if $D(\pm\nu_j, \mu)$ is calculated from Eq. (5-25), the running time is increased considerably. Fortunately, in many

TABLE III

Effect of the higher Legendre expansion coefficients of the scattering function f^{50} on the albedo α and transmittance β from a slab for which $c = 0.95$, $\tau = 1.0$ and $\mu_0 = 0.5$.[a]

l	M	α	β
50	4	0.04313901	0.70660650
40	4	0.04313901	0.70660650
30	4	0.04313901	0.70660650
15	4	0.043134	0.706611
10	4	0.04319	0.70664
8	4	0.0420	0.7079
6	3	0.0373	0.711
4	2	0.0462	0.697

[a] The index l denotes the number of coefficients retained, i.e., $b_n = 0$ if $n > l$.

problems this function may be calculated quite accurately from Eqs. (2-4) and (2-5) even for highly anisotropic scattering functions by simply ignoring the higher Legendre coefficients b_n. For the calculation of the albedo α and transmittance β, most of the higher coefficients may be ignored. In Table III the calculated values of α and β are seen to depend significantly only on the first few coefficients. Even the emergent distributions are not strongly dependent on the higher coefficients, a result seen from Fig. 2.

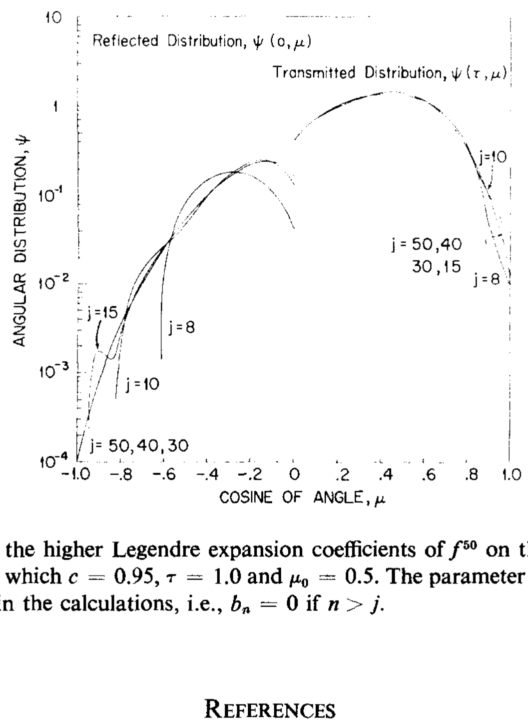

FIG. 2. Effect of the higher Legendre expansion coefficients of f^{50} on the emergent distributions from a slab for which $c = 0.95$, $\tau = 1.0$ and $\mu_0 = 0.5$. The parameter j denotes the number of coefficients used in the calculations, i.e., $b_n = 0$ if $n > j$.

REFERENCES

1. J. R. MIKA, *Nucl. Sci. Eng.* **11** (1961), 415.
2. J. K. SHULTIS AND H. G. KAPER, *Aston. Astrophys.* **3** (1969), 110.
3. S. CHANDRASEKHAR, "Radiative Transfer," Dover, New York, 1960.
4. D. F. MAYERS, *M.N.R.A.S.* **123** (1962), 471.
5. R. BELLMAN, H. KAGIWADA, R. KALABA, AND S. UENO, *J. Quant. Spectrosc. Radiat. Transfer* **6** (1966), 479.
6. S. PAHOR AND P. F. ZWEIFEL, *J. Math. Phys.* **10** (1969), 581.
7. T. W. MULLIKIN, *Amer. Math. Soc. Trans.* **113** (1964), 316.
8. J. K. SHULTIS, *Astron. Astrophys.* **14** (1971), 463.
9. H. R. KAPER, Report TW-37, Mathematisch Instituut, Univ. of Groningen, the Netherlands, 1966.

10. H. G. Kaper, J. K. Shultis and J. G. Veninga, Report TW-65, Mathematisch Instituut, Univ. of Groningen, the Netherlands, 1969.
11. H. G. Kaper, J. K. Shultis and J. G. Veninga, *J. Comp. Phys.* **6** (1970), 288.
12. E. Inonu and P. F. Zweifel, "Developments in Transport Theory," Academic Press, New York, 1967, 86.
13. W. W. Engle, Jr., Report K-1693, UCC Computing Tech. Ctr. 1967; also included as part of ORNL-RSIC Doc. CCC-82.